0.1 Chapter 3 – Electrostatics

0.1.1 Coulomb's Law

$$\vec{E}(r) = \frac{1}{4\pi\epsilon_0} \int \frac{\hat{r}}{r^2} \rho(r') d\tau' \tag{1}$$

0.1.2 Dipole Moment Definition

$$\vec{p} = \int \vec{r'} \rho(\vec{r'}) d\tau' \tag{2}$$

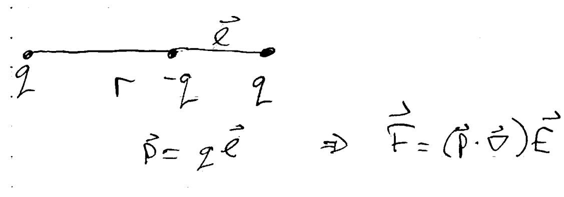

Figure 1: Geometry of a dipole

Force on a dipole due to a monopole:

$$\vec{p} = q\vec{l} \tag{3}$$
$$\vec{F} = (\vec{p} \cdot \vec{\nabla})\vec{E} \tag{4}$$
$$\tag{5}$$

where $\vec{l}$ is the direction vector from the negative to the positive pole.

Electric Dipoles (see Fig 1):

$$\vec{E} = \frac{P}{4\pi\epsilon_0 r^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta})$$

Torque on a dipole:

$$\vec{\tau} = \vec{p} \times \vec{E}$$

Potential associated with a dipole $\vec{p}$

$$V_{\text{dip}} = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}$$

0.1.3 Uniqueness Theorem

See figures 2 and 3.

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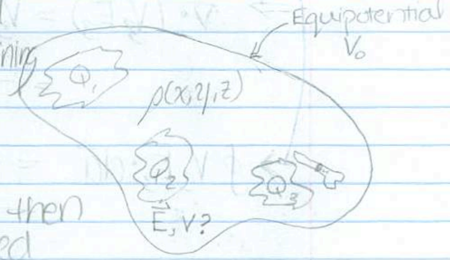
Second Uniqueness Theorem

In a volume surrounded by an equipotential, containing specified charge density $\rho(x,y,z)$ and containing individual conductors with total charges $Q_1, Q_2, Q_3, \ldots$, then the E field within the closed boundary is uniquely determined.



Proof

Assume two sol'ns: $\vec{E}_1, \vec{E}_2$

between conductors: $\vec{\nabla}\cdot\vec{E}_1 = \frac{\rho}{\varepsilon_0}$ $\quad \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon}$

boundaries around conductors:

$$\oint_{\text{conductor } i} \vec{E}_1\cdot d\vec{a} = \frac{Q_i}{\varepsilon_0} \qquad \oint_{\text{conductor } i} \vec{E}_2\cdot d\vec{a} = \frac{Q_i}{\varepsilon_0}$$

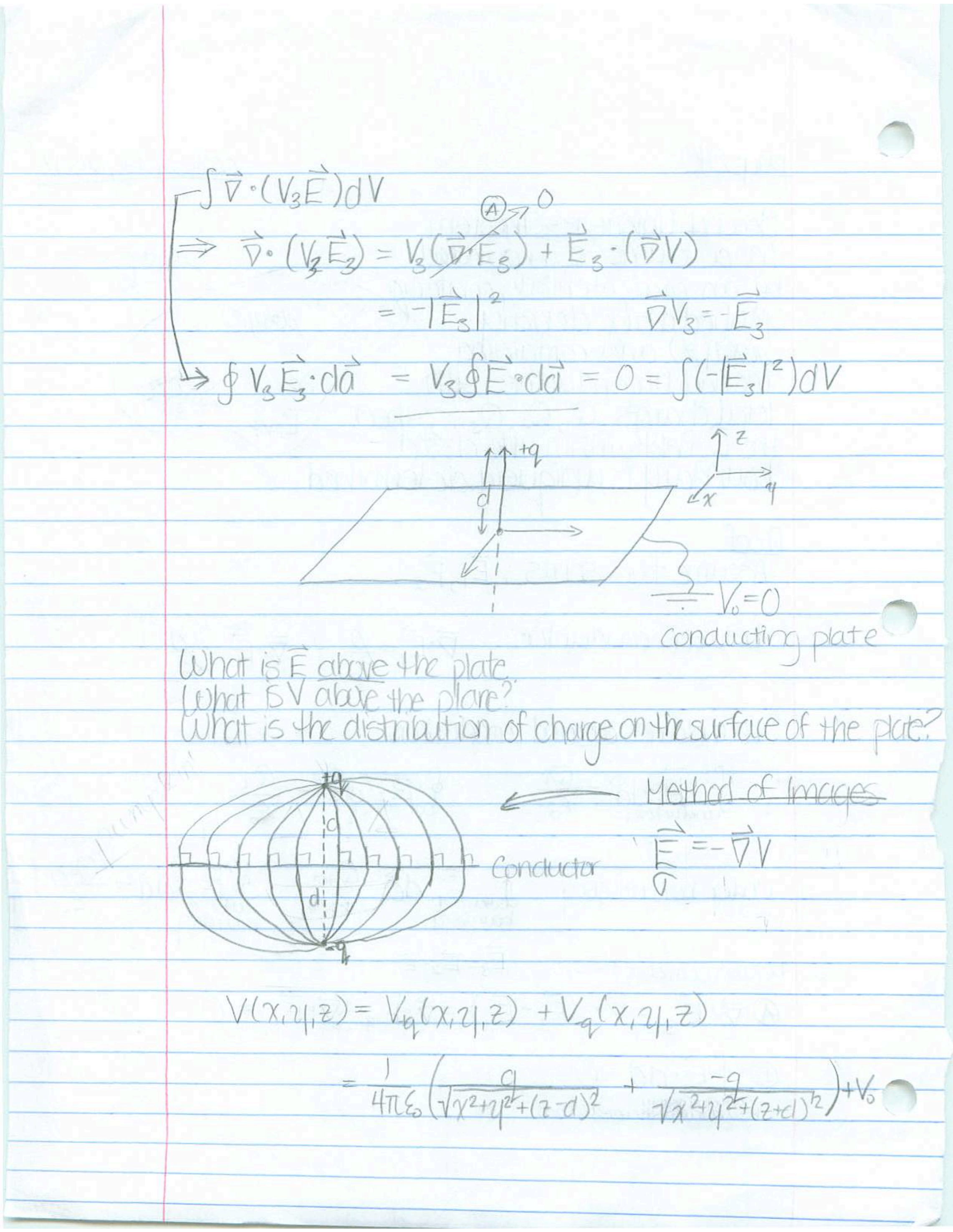
outer boundary: $\oint_{\text{outer boundary}} \vec{E}_1\cdot d\vec{a} = \frac{Q_{tot}}{\varepsilon_0}$ $\quad \oint_{\text{outer boundary}} \vec{E}_2\cdot d\vec{a} = \frac{Q_{tot}}{\varepsilon_0}$

between conductors: $\vec{E}_3 = \vec{E}_2 - \vec{E}_1$

(A) $\vec{\nabla}\cdot\vec{E}_3 = \vec{\nabla}\cdot(\vec{E}_2 - \vec{E}_1) = \vec{\nabla}\cdot\vec{E}_2 - \vec{\nabla}\cdot\vec{E}_1 = 0$

(B) $\oint_{\substack{\text{outer boundary} \\ \text{boundaries of the conductors}}} \vec{E}_3\cdot d\vec{a} = 0$

Figure 2: Uniqueness Theorem part 1



$$-\int \vec{\nabla}\cdot(V_3\vec{E})\,dV$$

$$\Rightarrow \vec{\nabla}\cdot(V_3\vec{E}_3) = V_3(\underbrace{\vec{\nabla}\cdot\vec{E}_3}_{\text{(A)} \to 0}) + \vec{E}_3\cdot(\vec{\nabla}V)$$

$$= -|\vec{E}_3|^2 \qquad \vec{\nabla}V_3 = -\vec{E}_3$$

$$\to \oint V_3\vec{E}_3\cdot d\vec{a} = V_3\oint \vec{E}\cdot d\vec{a} = 0 = \int(-|\vec{E}_3|^2)\,dV$$

What is $\vec{E}$ above the plate

What is V above the plate?

What is the distribution of charge on the surface of the plate?

~~Method of Images~~

$$\vec{E} = -\vec{\nabla}V$$

σ

$$V(x,y,z) = V_q(x,y,z) + V_{-q}(x,y,z)$$

$$= \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} + \frac{-q}{\sqrt{x^2+y^2+(z+d)^2}}\right) + V_0$$

Figure 3: Uniqueness Theorem part 2

0.1.4 Separation of Variables

In a region of no charge density $\rho = 0$ then the laplacian of the potential is zero $\vec{\nabla}^2 V = 0$.

Assume a solution

$$V(x, y, z) = A(x)B(y)C(z)$$

then

$$\begin{aligned} A''(x)B(y)C(z) + A(x)B''(y)C(z) + A(x)B(y)C''(y) &= 0 \quad (6) \\ \frac{A''(x)}{A(x)} + \frac{B''(y)}{B(y)} + \frac{C''(z)}{C(z)} &= 0 \quad (7) \\ C_1 + C_2 + C_3 &= 0 \quad (8) \end{aligned}$$

See Figures 4, 5 and 6.

0.2 ODE Solutions

$$\begin{aligned} \frac{d^2Y}{dy^2} = k^2Y \rightarrow R^2 = k^2 \quad (9) \\ y'' = R^2, y' = R, y = 1 \quad (10) \\ Y(y) = C_1e^{ky} + C_2e^{-ky} \quad (11) \end{aligned}$$

If you have $R = ik$ (i.e. $R^2 = -k^2$) then you use a general solution :

$$Y(y) = C_1 \sin(ky) + C_2 \cos(ky)$$

0.3 Chapter 5 – Magnetism

Force on a moving charge due to a magnetic field:

$$\begin{aligned} \vec{F}_{\text{mag}} = q(\vec{v} \times \vec{B}) \quad (12) \\ \vec{F} = Q(\vec{E} + (\vec{v} \times \vec{B})) \quad (13) \end{aligned}$$

Magnetic forces depending on the current geometry:

$$\begin{aligned} \vec{F} &= I \int (d\vec{l} \times \vec{B}) \quad \vec{I} = \lambda\vec{v} \quad (14) \\ &= \int (\vec{K} \times \vec{B}) da \quad \vec{K} = \sigma\vec{v} \quad (15) \\ &= \int (\vec{J} \times \vec{B}) dV \quad \vec{J} = \rho\vec{v} \quad (16) \end{aligned}$$

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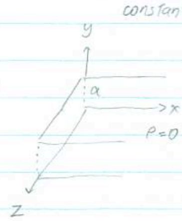
Laplace's Eq'n $\vec{\nabla}^2 V = \rho = 0$, how to find V?

If the geometry of the Boundary Potentials can be described by separate variables, then so the sol'n $V(x,y,z) = X_{(x)} Y_{(y)} Z_{(z)}$

$$\vec{\nabla}^2 V = YZ \frac{\partial^2}{\partial x^2} X_{(x)} + XZ \frac{\partial^2}{\partial y^2} Y_{(y)} + XY \frac{\partial^2}{\partial z^2} Z_{(z)}$$

$$\frac{1}{V} \vec{\nabla}^2 V = \underbrace{\frac{1}{X} \frac{\partial^2}{\partial x^2} X_{(x)}}_{\text{constant,}} + \underbrace{\frac{1}{Y} \frac{\partial^2}{\partial y^2} Y_{(y)}}_{C_2} + \underbrace{\frac{1}{Z} \frac{\partial^2}{\partial z^2} Z_{(z)}}_{C_3} = 0$$

which $C_1 + C_2 + C_3 = 0$



2- semiinfinite plane

① x-z plane $V=0$ at $y=0$, $y=a$

② $V=V_0$ when $x=0$

③ $V \to 0$ as $x \to \infty$

1. $\to$ $Z_{(z)} = 1$ 2. $\to$ $\vec{\nabla}^2 V = C_1 + C_2 = 0$

$$\frac{\partial^2}{\partial x^2} X_{(x)} = C_1 X_{(x)} \quad \to \quad C_1 = K^2 \qquad C_2 = -C_1 = -K^2$$

$$X_{(x)} = Ae^{kx} + Be^{-kx} \qquad Y_{(y)} = C\sin(ky) + D\cos(ky)$$

Boundary ① $x \to \infty$ $V \to 0$ so $A = 0$

② $V=0$ at $y = 0$ n' a so $D=0$ now make $B \times C = A$

now $Ae^{-kx}\sin(ka) = 0$ $K = \frac{n\pi}{a}$ $n \in \mathbb{Z}^+$ $\sum C_n \sin\frac{n\pi}{a}$

since $V(0, 0, \infty, z) = V_0$ $\sum C_n \sin\frac{n\pi}{a} = V_0$

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Figure 4: Separation of Variables part 1

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$$Y_{(y)} = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{a}y\right) = V_0$$

$$\int_0^a \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{a}y\right)\sin\left(\frac{n'\pi}{a}y\right) dy = \int_0^a V_0 \sin\left(\frac{n\pi}{a}y\right) dy$$

bring C_n out of integral

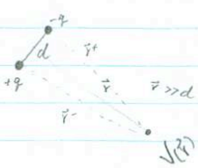
$$\int_0^a \sin\left(\frac{n\pi}{a}y\right)\sin\left(\frac{n'\pi}{a}y\right) dy = \begin{cases} 0 & \text{if } n \neq n' \\ \frac{a}{2} & \text{if } n = n' \end{cases} \rightarrow \frac{a}{2}\delta_{n,n'}$$

$$C_{n'} = \frac{2}{a}\int_0^a V_0 \sin\left(\frac{n'\pi}{a}y\right) dy = \frac{2V_0}{a} - \cos\left(\frac{n'\pi}{a}y\right)\frac{a}{n'\pi}\Big|_0^a$$

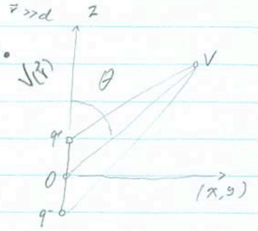
$$= \frac{2V_0}{n'\pi}\left[1 - \cos(n'\pi)\right] = \frac{4V_0}{n'\pi} \text{ for } n' = 1, 3, 5 \text{ and } 0 \text{ for } n = 0, 2, 4$$

$$V_{(x,y,z)} = \sum_{n=1,3,5}^{\infty} \frac{4V_0}{n\pi} e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)$$

The Dipole



$$V_{(\vec{r})} = \frac{1}{4\pi\varepsilon_0}\left(\frac{q^+}{r^+} + \frac{q^-}{r^-}\right) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r^+} - \frac{1}{r^-}\right)$$



must define a coordinate

then we have $V_{(r)} = \frac{q}{4\pi\varepsilon_0}\frac{d}{r^2}\cos\theta$

a dipole has a direction $|\vec{P}| = qd$

soo $\boxed{V_{(r)} = \frac{1}{4\pi\varepsilon_0}\vec{p}\cdot\hat{r}/r^2}$

$$\vec{E}_{(r)} = -\vec{\nabla}\vec{V}_{(r)} = -\left[\frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\hat{\phi}\right]\frac{1}{4\pi\varepsilon_0}\frac{P\cos\theta}{r^2}$$

$$= \frac{P}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

Figure 5: Separation of Variables part 2

ODE Solutions

$$A''(x) + \lambda A(x) = 0 \qquad \lambda \in \mathbb{R}$$

Case 1: $\lambda < 0$

$$A(x) = a\cosh(\sqrt{s}\,x) + b\sinh(\sqrt{s}\,x) \qquad (a,b) \in \mathbb{R}, \quad s = -\lambda$$

Case 2: $\lambda = 0$

$$A(x) = ax + b \qquad (a,b) \in \mathbb{R}$$

Case 3: $\lambda > 0$

$$A(x) = a\cos(\sqrt{\lambda}\,x) + b\sin(\sqrt{\lambda}\,x)$$

Figure 6: ODEs useful for the Uniqueness Theorem

Continuity equations :

$$\vec{\nabla}\cdot\vec{J} = -\frac{d\rho}{dt}$$

for magnetostatics $\vec{\nabla}\cdot\vec{J} = 0$.

0.3.1 Biot-Savart Law

$$\begin{aligned} \vec{B}(\vec{r}) &= \frac{\mu_0}{4\pi}\int \frac{\vec{I}\times\hat{r}}{|r|^2}dl' && (17)\\ &= \frac{\mu_0}{4\pi}\int \frac{\vec{K}(\vec{r}')\times\hat{r}}{|r|^2}da' && (18)\\ &= \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}')\times\hat{r}}{|r|^2}dV' && (19) \end{aligned}$$

0.4 Magnetic Maxwell's Equations

$$\begin{aligned} \vec{\nabla}\times\vec{B} &= \mu_0\vec{J} && (20)\\ \vec{\nabla}\cdot\vec{B} &= 0 && (21)\\ \int(\vec{\nabla}\times\vec{B})\cdot d\vec{a} &= \oint \vec{B}\cdot d\vec{l} && (22)\\ &= \mu_o\int \vec{J}\cdot d\vec{a} && (23)\\ \text{so } \oint \vec{B}\cdot d\vec{l} = \mu_0 I_{\text{enc}} &\ (\text{Ampere's Law}) && (24)\\ & && (25) \end{aligned}$$

0.4.1 Electric Maxwell's Equations

$$\begin{aligned} \int \vec{E}\cdot d\vec{a} &= \frac{Q_{\text{enc}}}{\epsilon}\ (\text{Gauss' Law}) && (26)\\ \vec{\nabla}\cdot\vec{E} &= \frac{\rho}{\epsilon_0} && (27)\\ \vec{\nabla}\times\vec{E} &= 0 && (28) \end{aligned}$$

Magnetostatic Vector Potential:

$$\begin{aligned} \vec{B} &= \vec{\nabla}\times\vec{A} && (29)\\ \vec{\nabla}\cdot\vec{A} &= 0 && (30)\\ \vec{\nabla}^2\vec{A} &= -\mu_0\vec{J} && (31) \end{aligned}$$

Definition of the Vector Potential:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,')}{|r|} dV' \tag{32}$$

$$= \frac{\mu_0}{4\pi}\int \frac{\vec{K}(\vec{r}\,')}{|r|} da' \tag{33}$$

$$= \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l}\,'}{|r|} \tag{34}$$

Dipoles:

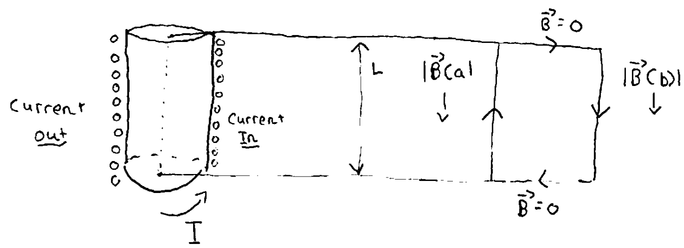
$$m = I\int da = I(\text{area}) \tag{35}$$

$$\vec{B}_{\text{dip}}(\vec{r}) = \frac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right) \tag{36}$$

0.4.2 Cyclotron Motion

$$QvB = \frac{mv^2}{\text{Radius}} \tag{37}$$

Infinite Solenoid $\quad \chi$ loops / unit length



$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$$

$(-B(a)L) + B(b)L = \mu_0 I_{enc} = 0$ (no current outside cylinder)

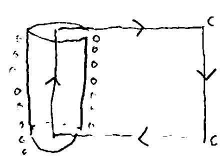
$$B(b)L - B(a)L = 0$$

$$\boxed{B(a) = B(L)}$$

Far away, the solenoid looks like a simple line of current. $B(\infty)=0$,

$\therefore\ \vec{B}=0$ outside.

Inside:



$$\vec{B}_{center}\cdot L = 0\cdot C + 0\cdot L + 0\cdot C = \mu_0 I_{enc} = \mu_0 \chi L I$$

$\therefore$

$\vec{B}_{center} = \mu_0 \chi I \Rightarrow$ everywher inside

Figure 7: Infinite Solonoid Problem

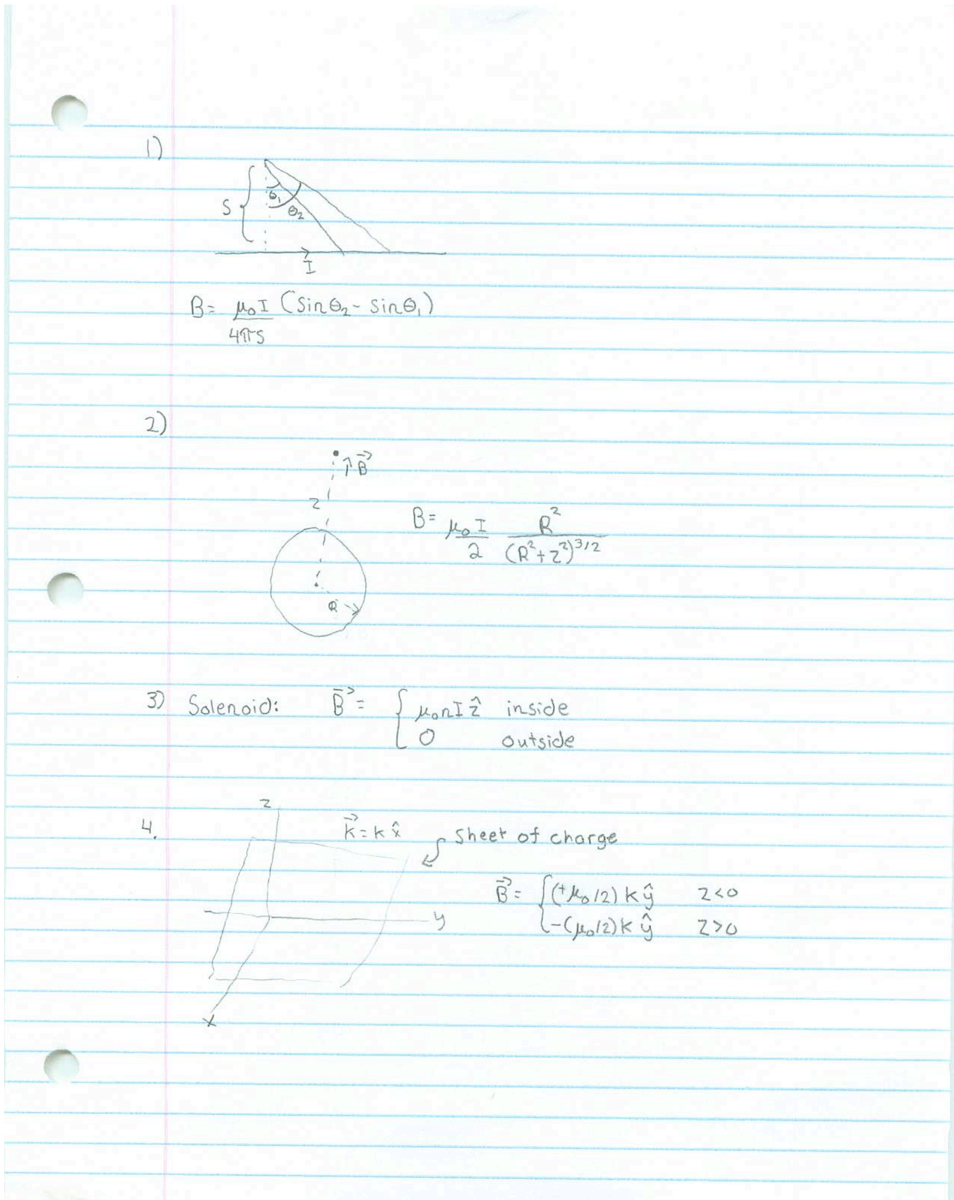


Figure 8: Current Geometries and B-field Solutions.