Faculty of Science
FINAL EXAMINATION

Physics - PHYS 340
Electricity and Magnetism

Friday, December 9th, 2005, 9 am-12 noon
Examiner: Prof. A. Cumming
Associate Examiner: Prof. R. Rutledge

INSTRUCTIONS

Answer all questions.

Section A is worth 25% of the total marks.

Answer in an exam book. You may keep the exam paper.

Credit will be given for attempting a solution, even if you do not reach the final answer. Make sure to show all your working. Remember to use symbols wherever possible, and insert numbers into formulae only at the end of the calculation.

Calculators are allowed.

This is a closed-book exam. No notes or texts allowed.

This exam comprises 5 pages, including this cover page.

TRANSLATION DICTIONARIES ARE ALLOWED
EXAM IS PRINTED DOUBLE-SIDED

SECTION A (Short answers)

1. Show that the dipole moment of a collection of charges does not depend on the choice of origin if the total charge is zero.

2. What is the dipole moment of a circular current loop with radius a carrying current I? Use the Biot-Savart law,

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int \frac{I d\boldsymbol{l}' \times (\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3},$$

to calculate the magnetic field on the symmetry axis of the loop. Show that it has the expected form for $r \gg a$.

3. Give an example of a non-uniform vector field which has zero-divergence. Comment on whether your example could be an electrostatic field.

4. The values of the vertical potential gradient at heights of 100 and 1000 m above the Earth's surface are measured to be 110 and 25 Vm^{-1} respectively. Assuming that the charged particles in the atmosphere are singly-ionized ions, what is the mean number of ions per cm^3? The permittivity of free space is $\epsilon_0 = 8.85 \times 10^{-12}\ \text{C}^2/\text{Nm}^2$. The charge of an electron is 1.6×10^{-19} C.

5. Estimate the resistance of the filament of a 100 W light bulb. Why does a light bulb usually fail at the moment when you switch the light on, rather than after the light has been on for a while?

SECTION B

6. (a) Write down Ampere's law in a magnetic material, carefully defining all quantities.

(b) A coaxial cable consists of a conducting cylinder of radius a surrounded by a conducting cylindrical shell with inner radius b and outer radius c. The space between the two conductors ($a < r < b$) is filled with a magnetic material with susceptibility χ. The two conductors carry equal but oppositely-directed currents I, uniformly distributed in each conductor. Calculate the magnetic field everywhere inside the cable, and draw a graph of magnetic field strength as a function of cylindrical radius r.

(c) Show that the inductance per unit length of the cable is

$$\frac{\mu_0}{2\pi}\left[\mu_r \ln\left(\frac{b}{a}\right) + \frac{c^2}{c^2 - b^2}\ln\left(\frac{c}{b}\right)\right].$$

Simplify the last term for the case when the outer cylinder is very thin.

7. (a) What are the conditions that must be satisfied by the electric potential V, electric field $\boldsymbol{E}$, and electric displacement $\boldsymbol{D}$ at the boundary between two dielectrics?

(b) A point charge q is a perpendicular distance d from a semi-infinite dielectric with dielectric constant ϵ_r. The surface of the dielectric is in the x, y plane at $z = 0$. This problem can be solved with the method of images, which gives

$$\begin{aligned} V_+(x,y,z) &= \frac{q}{4\pi\epsilon_0(\epsilon_r+1)}\left[\frac{\epsilon_r+1}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{\epsilon_r-1}{\sqrt{x^2+y^2+(z+d)^2}}\right] \\ V_-(x,y,z) &= \frac{q}{4\pi\epsilon_0(\epsilon_r+1)}\left[\frac{2}{\sqrt{x^2+y^2+(z-d)^2}}\right], \end{aligned}$$

where V_+ is the potential for $z > 0$ and V_- is the potential for $z < 0$. Show that this solution satisfies the appropriate boundary conditions at $z = 0$, and draw a diagram to illustrate the image charge configuration in each case.

(c) How much work has to be done to move the charge to infinity?

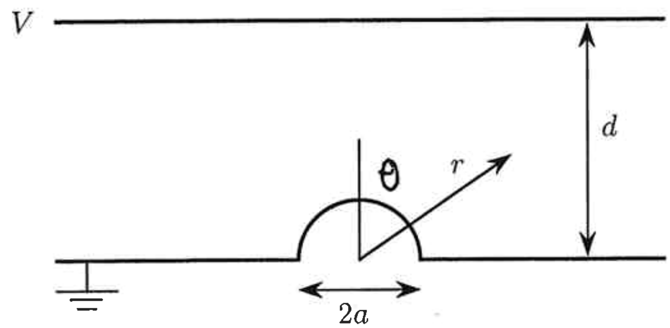
(d) In an experiment, a beam of electrons with velocity v_i must pass parallel to a dielectric plate whose length in the beam direction is L. What is the minimum perpendicular distance that must separate the beam and plate so that the electrons can pass the plate without striking it? Consider electrostatic forces only. You will need the integral

$$\int_d^0 dz\left[\frac{1}{z}-\frac{1}{d}\right]^{-1/2} = \frac{\pi d^{3/2}}{2}.$$

[Answer: $(e^2L^2\chi/2\pi^3\epsilon_0(\epsilon_r+1)m_e v_i^2)^{1/3}$, where e is the charge on an electron, and m_e is the electron mass.]

8. (a) Why are solutions to Laplace's equation with either the potential V or potential gradient $\hat{n}\cdot\nabla V$ specified on a closed boundary important in electrostatics problems with conductors? Show that $\cos\theta/r^2$ and $r\cos\theta$ are solutions to Laplace's equation in spherical polar coordinates.

(b) A capacitor consists of two large parallel plates separated by distance d, with a small hemispherical protrusion of radius a on the lower plate. The upper plate is held at potential V, and the lower plate is grounded. Assume that d is large compared to a, so that the field a large distance from the lower plate is uniform.



By writing the potential as a linear combination of the solutions $\cos\theta/r^2$ and $r\cos\theta$, show that the charge density σ on the lower plate is given by

$$\sigma = \frac{-3\epsilon_0 V}{d}\cos\theta$$

on the protrusion, and

$$\sigma = -\frac{\epsilon_0 V}{d}\left(1 - \frac{a^3}{r^3}\right)$$

on the flat part of the lower plate. Sketch the electric field lines.

(c) The maximum voltage that can be applied to the capacitor is reduced by the presence of the protrusion – by how much?

[In spherical polar coordinates,

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2},$$

and

$$\nabla = \left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right).$$

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