Faculty of Science
FINAL EXAMINATION

Physics - PHYS 340
Electricity and Magnetism

Wednesday, December 8th, 2004, 2pm-5pm
Examiner: Prof. A. Cumming
Associate Examiner: Prof. R. Rutledge

INSTRUCTIONS

Answer **all** *questions in section A, and* **either** *one question from section B and two questions from section C,* **or** *two questions from section B and one question from section C.*

Section A will be worth approximately a quarter of the total marks.

Answer in an exam book. You may keep the exam paper.

Credit will be given for attempting a solution, even if you do not reach the final answer. Make sure to show all your working. Remember to use symbols wherever possible, and insert numbers into formulae only at the end of the calculation.

Calculators are allowed.

This is a closed-book exam. No notes or texts allowed.

This exam comprises 7 pages, including this cover page.

FUNDAMENTAL CONSTANTS

$$c = 3.00 \times 10^{8} \text{ m/s}$$
$$\epsilon_0 = 8.85 \times 10^{-12} \text{ C}^2/\text{Nm}^2$$
$$\frac{1}{4\pi\epsilon_0} = 10^{-7}c^2 = 9 \times 10^{9} \text{ Nm}^2/\text{C}^2$$
$$\mu_0 = 4\pi \times 10^{-7} \text{ N/A}^2$$
$$e = 1.60 \times 10^{-19} \text{ C}$$

USEFUL FORMULAE

Biot-Savart Law

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int \frac{I d\boldsymbol{l}' \times (\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3}$$

Magnetic vector potential

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{J}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|}\, d^3\boldsymbol{r}'$$

Multipole expansion for electric potential

$$V(\boldsymbol{r}) \approx \frac{Q}{4\pi\epsilon_0 r} + \frac{\boldsymbol{p}\cdot\hat{\boldsymbol{r}}}{4\pi\epsilon_0 r^2} + \frac{\hat{\boldsymbol{r}}\cdot\boldsymbol{Q_2}\cdot\hat{\boldsymbol{r}}}{4\pi\epsilon_0 r^3} + \mathcal{O}\left(\frac{1}{r^4}\right)$$

Quadrupole moment of a collection of point charges

$$\boldsymbol{Q_2} = \sum_i \frac{q_i}{2}\left(3\boldsymbol{r}_i\boldsymbol{r}_i - |\boldsymbol{r}_i|^2\,\boldsymbol{I}\right)$$

Vector potential of a dipole

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0 \boldsymbol{m}\times\hat{\boldsymbol{r}}}{4\pi r^2}$$

VECTOR DERIVATIVES

Cylindrical coordinates (r, ϕ, z)

$$\begin{aligned}
d\boldsymbol{l} &= \hat{\boldsymbol{r}} dr + \hat{\boldsymbol{\phi}} r d\phi + \hat{\boldsymbol{z}} dz, \qquad dV = r dr d\phi dz \\
\boldsymbol{\nabla} f &= \hat{\boldsymbol{r}} \frac{\partial f}{\partial r} + \hat{\boldsymbol{\phi}} \frac{1}{r} \frac{\partial f}{\partial \phi} + \hat{\boldsymbol{z}} \frac{\partial f}{\partial z} \\
\boldsymbol{\nabla} \cdot \boldsymbol{A} &= \frac{1}{r} \frac{\partial}{\partial r}(r A_r) + \frac{1}{r} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \\
\boldsymbol{\nabla} \times \boldsymbol{A} &= \hat{\boldsymbol{r}} \left(\frac{1}{r} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) + \hat{\boldsymbol{\phi}} \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) + \hat{\boldsymbol{z}} \left[\frac{1}{r} \frac{\partial}{\partial r} (r A_\phi) - \frac{1}{r} \frac{\partial A_r}{\partial \phi} \right] \\
\nabla^2 f &= \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}
\end{aligned}$$

Spherical coordinates (r, θ, ϕ)

$$\begin{aligned}
d\boldsymbol{l} &= \hat{\boldsymbol{r}} dr + \hat{\boldsymbol{\theta}} r d\theta + \hat{\boldsymbol{\phi}} r \sin\theta d\phi, \qquad dV = r^2 \sin\theta dr d\theta d\phi \\
\boldsymbol{\nabla} f &= \hat{\boldsymbol{r}} \frac{\partial f}{\partial r} + \hat{\boldsymbol{\theta}} \frac{1}{r} \frac{\partial f}{\partial \theta} + \hat{\boldsymbol{\phi}} \frac{1}{r \sin\theta} \frac{\partial f}{\partial \phi} \\
\boldsymbol{\nabla} \cdot \boldsymbol{A} &= \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta}(\sin\theta A_\theta) + \frac{1}{r \sin\theta} \frac{\partial A_\phi}{\partial \phi} \\
\boldsymbol{\nabla} \times \boldsymbol{A} &= \frac{\hat{\boldsymbol{r}}}{r \sin\theta} \left[\frac{\partial}{\partial \theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial \phi} \right] + \frac{\hat{\boldsymbol{\theta}}}{r} \left[\frac{1}{\sin\theta} \frac{\partial A_r}{\partial \phi} - \frac{\partial}{\partial r}(r A_\phi) \right] + \\
&\quad \frac{\hat{\boldsymbol{\phi}}}{r} \left[\frac{\partial}{\partial r} (r A_\theta) - \frac{\partial A_r}{\partial \theta} \right] \\
\nabla^2 f &= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 f}{\partial \phi^2}
\end{aligned}$$

SECTION A

Short answers. Relevant formulae may be assumed without derivation.

1. Mylar is a dielectric material with $\epsilon_r = 3.0$. It is desired to construct a 100pF capacitor using a mylar spacer 10^{-4} m thick placed between two metal plates. How large an area is required for the metal plates?

2. Estimate the smallest radius of curvature that can be used for a conductor at 10^5 V, if the breakdown electric field strength of air is 3×10^6 V m^{-1}.

3. A 10 m length of wire lies horizontally on a table in the east-west direction. What voltage should be applied, and in which direction, to make the wire rise from the surface? (The wire has resistivity 2×10^{-8} Ω m and density 10^4 kg m^{-3}. The horizontal component of Earth's magnetic field is 1.8×10^{-5} T, and the acceleration due to gravity is $g = 9.8$ N kg^{-1}).

4. Consider the magnetic field $\boldsymbol{B} = axy\hat{\boldsymbol{x}} + by^2\hat{\boldsymbol{y}}$. What is the relation between the constants a and b? What current density $\boldsymbol{J}$ produces this field? Sketch the current distribution.

5. Electric power transmission lines operate at high voltage. Why?

SECTION B

6. (a) Write down Maxwell's equations for electrostatic fields in differential form. Use Stoke's theorem and the divergence theorem to derive the integral form of these equations. What does each equation tell you about the properties of the electric field?

(b) A capacitor is constructed from two concentric metal cylinders. The radius of the inner cylinder is a, the radius of the outer cylinder is b. The space between the cylinders is filled with air for which $\epsilon_r = 1.0$. What is the capacitance per unit length of this device?

(c) This capacitor, whose length is 10 cm, is placed upright in a dish of oil, which has $\epsilon_r = 3.0$, so that the space between the cylindrical electrodes is filled with oil to a depth of 5 cm. What is the capacitance of this configuration if the radii are $a = 5$ cm and $b = 6$ cm?

(d) The capacitor is now charged to a potential difference of 1000 V. How high will the oil rise between the capacitor electrodes if the density of oil is 800 kg m^{-3}?

7. (a) State the uniqueness theorem for electrostatic fields, and explain how it leads to the method of images.

(b) A point charge q is placed a distance d away from the centre of a grounded, conducting sphere of radius a (where $a < d$). The image charge for this case is a point charge $-aq/d$, distance a^2/d from the centre of the sphere. Write down the potential $V(r, \theta)$ for a point (r, θ) outside the conducting sphere, and show that it obeys the appropriate boundary conditions.

(c) Calculate the force on the charge, and the surface charge density on the sphere $\sigma(\theta)$.

(d) A second point charge $-q$ is placed on the opposite side of the sphere to the first charge, but also a distance d from the centre of the sphere. What are the image charges in this case? By considering the limit $d \rightarrow \infty$, but with q/d^2 held constant, derive the polarizability of the conducting sphere. (Recall that the polarizability α is the ratio of the induced dipole moment to the applied electric field.)

8. A small drop of oil is characterized by a relative dielectric constant $\epsilon_r = 1.5$ and a density of $\rho = 800$ kg m^{-3}; its radius is $R = 10^{-4}$ m. It is placed between capacitor plates which are parallel and which are separated by 1 cm. The oil drop is uncharged. A potential difference of 100 volts is placed across the capacitor plates. The dielectric constant of air may be taken to be $\epsilon_r = 1$.

(a) The general solution to Laplace's equation in spherical geometry is

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + B_l r^{-l-1} \right) P_l(\cos\theta).$$

Use this to derive the potential inside and outside the oil drop. You will need to use $P_1(\cos\theta) = \cos\theta$, and apply the appropriate boundary conditions at the surface of the oil drop and at a large distance from the oil drop.

(b) Calculate the distribution of bound charges on the surface of the drop.

(c) What is the value of the dipole moment induced on the drop?

(d) How large a field gradient would be required to suspend the drop in the gravitational field? (The acceleration due to gravity is $g = 9.8$ N kg^{-1}).

SECTION C

9. (a) Use the Biot-Savart law to find the magnetic field along the symmetry axis of a circular current loop with radius a.

(b) What is B in the limit $z \gg a$, where z is the distance along the symmetry axis? Is this what you expect, and if so why?

(c) A total charge of 10 C is spread uniformly over a circular insulating plate of radius 12 cm. The plate is then set spinning about its symmetry axis at a frequency of 50 Hz. What is the value of the magnetic field at its centre?

(d) What is the magnetic dipole moment of the spinning disk? Using this, write down the field at position (r, θ) far from the spinning disk.

10. (a) Write down the relations between $\boldsymbol{H}$, $\boldsymbol{B}$, and $\boldsymbol{M}$ for a general magnetic material, and for a linear magnetic material.

(b) Two needles are placed in a uniform magnetic field. One is made from diamagnetic material, one from paramagnetic material. Describe how each material responds to the applied magnetic field, and explain the microphysics that determines the response.

(c) Which way does each needle orient itself with respect to the magnetic field, and why?

(d) Two thin-walled long coaxial cylinders with radii a and b carry equal but opposite currents $\pm I$ parallel to the symmetry axis. The surface current densities are $I/2\pi a$ and $-I/2\pi b$. Between the cylinders is a material with magnetic susceptibility χ_m. Determine the field $\boldsymbol{H}(r,\phi)$, and magnetization $\boldsymbol{M}(r,\phi)$. Plot $H_\phi(r)$, and sketch the bound currents.

11. (a) State Faraday's law and Lenz's law.

(b) A laboratory demonstration of electromagnetic induction is to drop a bar magnet into a vertical length of copper pipe. It falls much more slowly than when dropped into a plastic pipe. Why? In your answer, you should sketch a graph of the applied magnetic field at a fixed position on the pipe wall as the bar magnet falls, the induced currents, and the resulting magnetic field and its effect on the bar magnet.

(c) A line charge λ is glued onto the rim of a wheel of insulating material of radius a, which is then suspended horizontally, so that it is free to rotate. Initially, the wheel is stationary, and there is a uniform vertical magnetic field $\boldsymbol{B}$ pointing straight up. Now someone turns the field off. The wheel begins to rotate. Why, and in which direction?

(d) Calculate the final angular momentum of the wheel.