

# Hysteresis

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## Abstract

This experiment investigated the hysteresis curve of a toroidal iron ring and found the energy required to bring the material around a hysteresis loop for a given maximum magnetic field. By varying the resistance in the circuit, the relation between magnetic flux density,  $B$ , and magnetic field in the ring,  $H$ , was observed. The graphs of the hysteresis curves  $B$  vs  $H$  show the lagging effect due to a changing magnetizing force. The area enclosed by these curves is equal to the work per unit volume. The plot of total work versus voltage indicated a linear relationship with slope  $9[1] \times 10^{-6} JV^{-1}$  and  $y$  intercept  $0.00011[1]J$ . The Steinmetz coefficient was calculated for each trial. These were found to be constant within error at  $56800[500]$ .

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## 1 Introduction

A demagnetized metal ring can easily be magnetized by the presence of an external magnetic field. If we were to plot the magnetization of the ring,  $B$ , vs the applied magnetic field,  $H$ , we would quickly see that the metal becomes saturated as in Figure ?? . Let us suppose that having magnetized the ring, we now wish to demagnetize it.

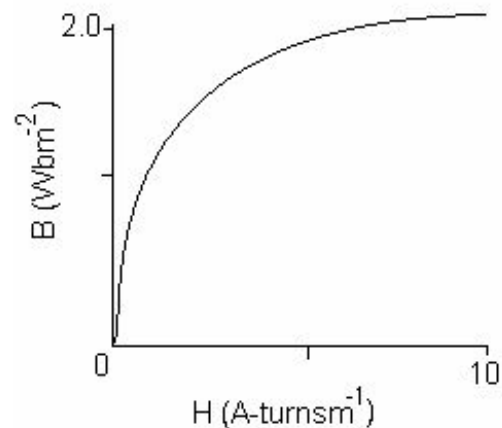


Figure 1: Magnetization of Soft Iron [?]

If the metal ring happens to be made of a ferromagnetic material, like Iron, the magnetization will in fact be permanent, meaning that the sample will remain magnetized even as the external magnetic field is brought back to zero. This is not what we would have expected; it is this phenomenon that is known as hysteresis, which can be described by noting that  $B$  lags behind the value of  $H$ .

There are several reasons that the hysteresis loop is important. Firstly, the loop is unique to the metal that the ring is composed of. In fact, hysteresis was once a method used to determine the identity of unknown metals and alloys. However, a more practical application of the

hysteresis data's uniqueness is to determine what kind of situation a magnet made from a given substance is most suited for. A very narrow hysteresis curve indicates that a given substance is a temporary or soft magnet, while fatter curves mean a more permanent, hard magnet. The difference between these two types of magnets is that a soft magnet will respond to a small magnetic field, while a hard magnet is permanent and will remain magnetized after a strong magnetic field has been applied and then removed. Soft magnets are often used in circuits found in places like motors, generators and transformers where energy loss due to magnetization should be minimized. Magnetic materials which lie between hard and soft are used a lot in recording devices where the stability of a hard magnet is required so that data is not lost, but a strong magnetic field must be able to erase the previously recorded data if desired.

The hysteresis loop is a signature of a magnetic material, and is a plot of magnetization against applied magnetic field. Slender loops indicate temporary, or soft magnets, while fatter loops indicate more permanent, hard magnets. In this experiment, the magnetic ring was initially unmagnetized. With a current send through the ring, it produces

The coercivity of a magnet is the magnetic field required to reduce its magnetization to zero. In 1743, Bernoulli developed the steel bar and horseshoe-shaped magnets whose shape increased their coercivity, but modern magnetic materials have a coercivity greater than their saturation magnetization and therefore resist self-demagnetization in any shape. The strength of a permanent magnet is given by its maximum energy product, twice the energy stored in the magnetic field around a magnet of optimum shape. This value doubled every twelve years during the 20th century, but current technology is at the point where another double would be difficult.

Hysteresis is used to describe the behavior of something that lags behind the cause that effects it. For example, when a paper is cumpled it slowly returns towards its original form once released but never completely regains it. When magnetized, the iron ring shows this type of behavior as its magnetization lags behind the magnetizing field applied. When the field is removed, the material retains some of its magnetization. The area of the hysteresis loop is a measure of the work required to overcome the this molecular magnetic friction. A larger area shows that a material behaves as more of a permanent magnet because it resists changing its magnetic alignment. The linear relationship between voltage and work suggests that

the energy required to overcome the resistance of the iron core to changing polarity is proportional to the strength of the magnetic field applied. The strength of the field induced in the iron that opposes the magnetizing field increases proportionally to its strength.

## 2 Theory

When a current is sent through the magnetizing coil, a magnetizing field,  $H$  is created in the iron and the ring is magnetized. This gives rise to a magnetic flux  $\phi$  (in Webers,  $Wb$ ) in the ring, which has density  $B$  (in Teslas,  $T$ ). Changes in the current cause a change in  $B$ . A second coil (search coil) is wound around the magnetizing coil and connected to a galvanometer. When  $B$  changes, a momentary current is created in the search coil, which passes through the galvanometer and is directly proportional to the deflection in the galvanometer.  $H$  (in Ampere-turns per meter) is calculated using

$$H = \frac{NI}{l} \quad (1)$$

where  $N$  is the number of turns in the magnetizing coil,  $I$  is the current (in Amperes,  $A$ ) and  $l$  is the circumference of the iron ring ( $m$ ).  $B$  is calculated using the equation,

$$B = \frac{N_s \phi_s d}{n A d_s} \quad (2)$$

here  $N_s \phi_s$  is the calibrated value for the Hibbert standard (in this case the value is  $1.72mWb$ ),  $d$  is the galvanometer reading ( $m$ ),  $n$  is the number of turns on the search coil,  $A$  is the area of the search coil ( $m$ ) and  $d_s$  is the initial half-amplitude reading of the galvanometer ( $m$ ).

To obtain a hysteresis loop, the magnetic field,  $H$ , is increased to saturation, then decreased to zero, then increased to saturation in the reverse polarity, then brought back to zero and finally fully re-magnetized in the original direction.  $H$  is increased and decreased gradually (by increasing and decreasing the resistance in the circuit) and the deflection of the galvanometer needle (used to calculate  $B$ ) is recorded with every change in  $H$ . When  $B$  is plotted against  $H$ , it is found that after the first magnetization,  $B$  lags behind  $H$ . This is hysteresis and is the result of a certain permanent magnetism in the iron. The area enclosed by the  $B-H$  curve represents the work per unit volume.

The work done throughout the whole process can be found by,

Ballistic Galvanometer-Wall-mounted, Leeds & Northrup (#11)  
 Damping Key (#7)  
 Hibbert standard flux – S2H (#6)  
 Plug Resistance Box (#2)  
 Resistance board of 14 resistors in series (#4)  
 Rowland Ring, cast iron  $F - C$  (#4)  
 Simpson DC Ammeter, Model 19, 0 – 1A  
 Power Supply, 0 – 24V (#9) trials 1,2,3 (#1) trials 4,5,6,7,8,9  
 Reversing Switch  $S - R$  (S8H)

Table 1: Equipment

$$W = AV \quad (3)$$

where  $A$  is the area of the hysteresis loop ( $AWbm^{-3}$ ) and  $V$  is the volume of the ring ( $m^3$ ). The loss of energy (the work) is due to molecular magnetic friction in the ring.

A relationship, discovered by Charles Steinmetz, exists between the area of the loop and the maximum applied magnetic flux density. The relationship is,

$$A = \eta \cdot (B_{max})^{1.6} \quad (4)$$

where  $A$  is the area of the  $B-H$  curve in  $AWbm^{-3}$ ,  $B_{max}$  is the maximum value of the flux density ( $T$ ), and  $\eta$  is the Steinmetz coefficient. The Steinmetz coefficient is unitless, because it represents a strictly empirical relationship.

With a current send through the ring, a magnetizing field,  $H$ , which magnetizes the iron. This results in a high flux density,  $B$ , in the ring. Changes in this current cause a momentary current in a second coil around the ring which is connected to a galvanometre used to measure the flux density.

### 3 Apparatus

## 4 Procedure

### 4.1 Demagnetizing the Coil

Hook up the coil to the demagnetizer between every trial, according to Figure ???. Start at the left of the rheostat, the coil is demagnetized when it has reached the right. This should take about ten minutes.

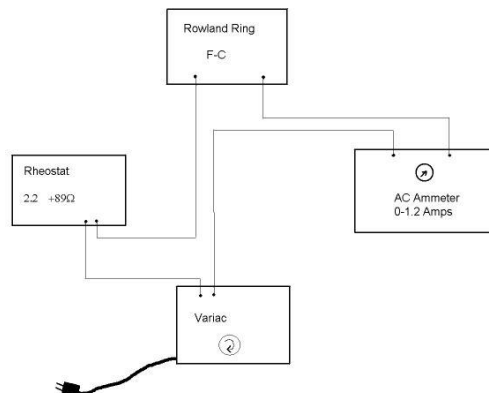


Figure 2: Demagnetizing Circuit

## 4.2 Setup and Trials

Wire the circuit according to Figure ???. Once the galvanometer has settled, adjust it to zero. Drop the Hibbert standard and record the displacement. Always include information on the sense of the deflection as the sign of the data. On our equipment, this was around  $12cm$ . This measurement is the half-amplitude reading of the galvanometer. With the reversing switch set to OPEN and all the switches on the resistance board open, the galvanometer should be at zero.

Turn on the power supply to 12V. Set the reversing switch to DIRECT and immediately record the current and deflection. One by one, close the switches from left to right, recording the current and deflection for each resistor. This is point **a** (Figure ??). One by one, open the switches from right to left, recording each measurement. With all the switches open, set the reversing switch to OPEN and record to deflection (the current is zero). This is point **b** (Figure ??).

Taking current and deflection reading between each step, set the reversing switch to REVERSE and successively close all the resistance switches from left to right. When they are all closed, this is point **c** (Figure ??). Now open the switches from right to left. Set the reversing switch to OPEN. This is point **d** (Figure ??).

Set the reversing switch to DIRECT then close all the switches from left to right. This point should be close to point **a** (Figure ??).

Repeat steps 1, 2 and 3 for the following voltages: 6, 7, 9, 10.5, 12, and 13.5 volts.

Calibrated Hibbert Standard	$N_s \phi_s$	1.72	$mWb$
Number of Turns Search Coil	$n$	24	
Diameter of the Ring	$d$	0.16	$m$
Area of Search Coil	$A$	0.020	$m^2$
Ballistic Constant	$kb$	$27.85 \pm 0.05$	$m^2 T^{-1}$
No. of Turns on Rolland Ring	$N$	2385	
Circumference of Ring	$l$	0.50265	$m$
Volume of Ring	$V$	0.0000503	$m^3$

Table 2: Equipment Data

### 4.3 Plotting Graphs

For each trial, use the half-amplitude displacement to calculate  $k_b$  and use it, along with the data collected to calculate and plot  $B$  versus  $H$ . The displacement used in calculating  $B$  is cumulative so each displacement measurement is added to the sum of all the preceding measurements, taking the sign of each reading into account. The area enclosed in the Hysteresis loop represents the energy lost per unit volume.

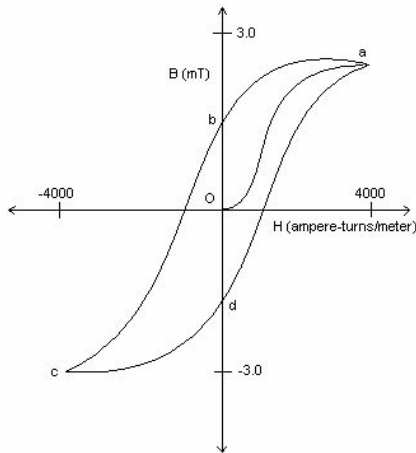


Figure 3: Sketch of Hysteresis Loop

### 4.4 Data and Analysis

The data in Table ?? summarizes the values of various properties of the equipment. These values were all used in order calculate  $B$  and  $H$  from the measured current and deflection. (See Appendix 2 for sample calculations.)

The extensive data and all the plots of  $B$  vs.  $H$  (hysteresis loops) can be found in Appendix 1 (Tables A1–A7 and graphs A1–A7). See Appendix 3 for the raw data. As a prototypical sample, the plot for the 10.10V trial can be seen below.

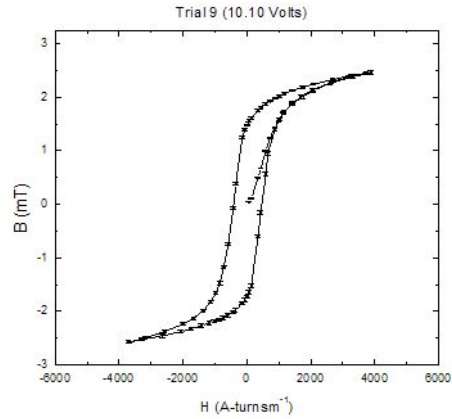


Figure 4: Trial 9 (10.10V)

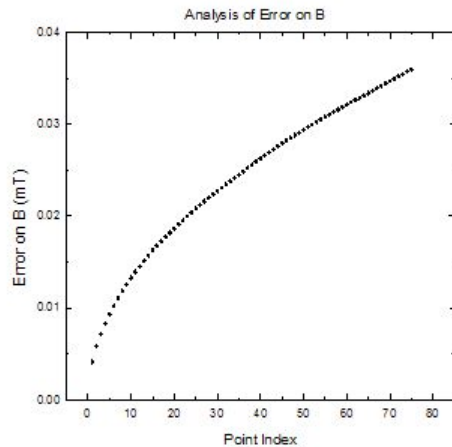


Figure 5: Analysis of Error on B (Trial 9)

The values of  $B$  are based on the sum of the previous deflection measurements, thus the error on each successive point on the plots grows. (See ?? and also see sample calculations in Appendix 2 for a detailed explanation of how this error is calculated.) This effect is shown in Figure ?? by the increasing size of the error bars around the

Trial	Voltage (V)	Area ( $AWbm^{-1}$ )	$\sigma_{Area}$	Work (J)	Voltage ( $\sigma_{Work}$ )	Area ( $AWbm^{-3}$ )	$B_{max}$ (T)	Steinmetz's constant	$\sigma_{Stein}$
1	11.89	4.3	0.2	0.0002141189	0.000008	4.2574	0.00265	57000	
2	9.00	3.81	0.14	0.00019190	0.000007	3.8062	0.00252	55000	
3	6.02	3.2	0.1	0.00016260	0.000005	3.2146	0.00220	58000	
5	10.48	4.17	0.15	0.0002101048	0.000008	4.1725	0.00258	58000	
6	13.50	4.5	0.2	0.0002271350	0.000009	4.5207	0.00275	57000	
8	7.00	3.4	0.1	0.00017270	0.000005	3.4168	0.00234	55500	
9	10.10	3.9	0.1	0.0001971000	0.000006	3.9205	0.00245	59000	

Table 3: Area and Work Data

Table 4: Steinmetz's Constant

hysteresis loop. The integrals of the hysteresis loops (area of the loops) were needed to find the work. These values were calculated by interpolating between points with a cubic spline and connecting the beginning and end of the loop with a line segment to form a closed curve. Details can be found in Appendix 2. These areas are multiplied by the volume of the ring to find work (Equation ??), and the results are summarized in Table ??.

The calculations required for determining the error on the area are very detailed and can be found in Appendix 2. Uncertainty in calculating area is due partly to the discrepancies of the endpoints. Since six of the seven used trials yielded graphs where the endpoints did not meet, the additional area had to be accounted for. Some graphs crossed over each other, leaving a closed loop and an additional open-ended area, while the other graphs left a single open loop. We drew a straight line joining point a (Figure ??) to the final point in every trial, then took the total enclosed area.

The error on the work depends on the error on the area (see Appendix 2). The plot of the work vs voltage is presented in Figure ?. This graph shows a linear relationship between the voltage and the work with all the points within experimental error of the linear least-squares fit. Once the area of each curve was found, the Steinmetz constant could be calculated using Equation ?. The results are summarized in Table ??.

The average value for the Steinmetz constant was found to be  $56800 \pm 500$ , and it is represented by the straight line on Graph ?. The error value is based on a weighted average of the data; see Appendix ?? for details.

## 5 Discussion

Hysteresis is used to describe the behavior of something that lags behind the cause that affects it. For example,

when a paper is crumpled it slowly returns towards its original form but doesn't completely regain it once released. When magnetized, the iron ring shows this type of behavior as its magnetization lags behind the magnetizing field applied. When the field is removed, the material retains some of its magnetization. The area of the hysteresis loop is a measure of the work required to overcome the molecular magnetic friction. A larger area shows that a material behaves as more of a permanent magnet because it resists changing its magnetic alignment. The linear relationship between voltage and work suggests that the energy required to overcome the resistance of the iron core to changing polarity is proportional to the strength of the magnetic field applied. The strength of the field induced in the iron that opposes the magnetizing field increases proportionally to its strength.

There were two main difficulties encountered when taking measurements. The first was in reading the galvanometer. Since the measurement is viewed, it is dependent on the person who reads it. The first difficulty here is that it is a moving scale which therefore needs to be read at the right time. The other is that there is a discrepancy between different people in reading very small numbers (what some people would read as four others may read as five, as well as a probable psychological tendency for one to read certain numbers more often than others).

The second difficulty encountered was reading the ammeter. The current reading should be taken immediately after the switch has been opened or closed because the current values jump, then continue to climb a few more seconds. The reason this is difficult is because sometimes the reading changes very quickly while the digital meter displays the current at discrete intervals. It was therefore difficult to be certain that the readings were taken at the right time. In order to get accurate readings, we would need a method of reading the first current displayed, before it changes, such as a graphical meter that plotted the current. In order to minimize this random error we always had the same person reading the current although this contributed towards more systematic error. Another source of

systematic error was the equipment we used. We had to assume the values given were absolute since no error was given. It is also possible that the galvanometer or the Hibbert flux standard were not calibrated. Since they are quite old, the galvanometer may not have been producing accurate deflections. In order to ensure that the equipment was accurate, we tested the resistance box to ensure it was  $130\Omega$  as indicated but we could also have tested the voltmeter and ammeter. One large source of error would be the mislabeling of the flux standard. While it was labeled as S2H which corresponded with a value of  $1.72mWb$  as our value for  $N_s\varphi_s$ . A different value here would have a significant effect on all of our calculated values.

Due to a fluctuating voltage source, we had to change our power supply after the first three trials. We do not know how constant it was during these trials and therefore this may account for some error in these trials.

## 6 Conclusion

This experiment measured the magnetization curve of an iron coil – known as a hysteresis loop – and analyzed its shape and properties. When a magnetic field is applied to the coil, it causes it to become magnetized to a saturation point; the polarity of the current is then reversed twice, magnetizing the iron to saturation in opposite polarities each time and thereby completing the loop. The area of this loop is equal to the work done per unit volume of the coil, indicating the loss of energy due to molecular magnetic friction. Analysis showed that change in the work was proportional to change in the voltage with a coefficient of  $9[1] \times 10^{-6}JV^{-1}$ . The fit line of work vs voltage should pass through zero since no voltage implies no work; instead our fit had an intercept of  $0.00011[1]J$ . This was caused by systematic error because the fit line passed cleanly through the points within the calculated random error. Possible explanations are additional energy loss not caused by hysteresis, small equipment miscalibrations, and ambient magnetic fields. Also, the endpoints of our loops were not cleanly defined – the most probable significant source of systematic error. We ran the experiment to a certain resistance and assumed that this accomplished the magnetic saturation, when in fact our high readings and crossing loops suggest that perhaps this may have passed the saturation point. In theory this would not affect the area, but in practice the lines after the saturation point do not coincide, leading to a consistently high area.

There are several points where the results from this experiment could be improved. The galvanometer should have been read to one tenth of a millimeter every reading. Equipment could have been calibrated against a known

standard to ensure accuracy. Original values read from the equipment were taken without error; some of these values might have been remeasured. Because the error on the measured deflection was cumulative, accuracy decreased with more measurements. On the other hand, more measurements gave a greater resolution to the curve. The experiment could be repeated once with a fewer, optimal number of points to give the most accurate result, then again with more, smaller resistance changes to give a more precise shape of the curve. Combining these two results would then achieve greater accuracy and precision.

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This experiment measured the magnetization curve of an iron coil – known as a hysteresis loop – and analyzed its shape and properties. When a magnetic field is applied to the coil, it causes it to become magnetized to a saturation point; the polarity of the current is then reversed and the coil magnetized to saturation in the opposite polarity. After being reversed again, the coil returns to saturation in the original polarity, completing the loop. With voltages of the order of  $10V$  and a resistance of  $130\Omega$ , the loops showed a magnetic saturation magnitude of the order of  $2.5mT$  at a maximum magnetizing field magnitude of the order of  $4000Am^{-1}$ . From this information, a Steinmetz coefficient of  $56800 \pm 500$  was found for the iron ring.

The area of this loop is equal to the work done per unit volume of the coil, indicating the loss of energy due to molecular magnetic friction. Analysis showed that change in the work was proportional to change in the voltage with a coefficient of  $8.89 \times 10^{-6}JV^{-1}$ . We expect that the fit line of work vs. voltage should pass through zero since with zero voltage there is no work done; instead our fit had an intercept of  $0.0001J$ . It is expected that this is caused by a systematic error because the fit line passed cleanly through the points within the calculated random error but was offset by this amount. Due to the small values, this error could be caused by small equipment miscalibrations or even ambient magnetic fields. Also, the endpoints of our loops were not cleanly defined, the most probable significant source of systematic error. We ran the experiment to a certain resistance and assumed that this accomplished the magnetic saturation, when in fact our high readings and crossing loops suggest that perhaps this may have passed the saturation point. In theory this would not affect the area, but in practice the lines after the saturation point do not coincide, leading to a consistently high area.

There are several points where the results from this experiment could be improved. The galvanometer should have been read to one tenth of a millimetre every reading, rather than only the first set. Equipment, specifically the galvanometer, could have been calibrated against a known

standard to ensure accuracy. Original values read from the equipment were taken without error; since the equipment was old, some of these values might have been remeasured. Because the error on the measured deflection was cumulative, accuracy decreased with more measurements. On the other hand, more measurements gave a greater resolution to the curve. The experiment could be repeated once with a fewer, optimal number of points to give an accurate result, then again with more, smaller resistance changes to give the more precise shape of the curve. Combining these two results might achieve more accuracy and precision to the loop.

## **7 Acknowledgements**

We would like to thank Mrs. E. Engelberg for her continuous aid and advice, Mr. Mario Della Neve for his help with the apparatus as well as Prof. M. Hilke for his advice on the analysis. A special thanks to Mrs. Engelberg's number crunchers for answering questions about Steinmetz coefficient units.

## A Sample Calculations

Trial 1 calculations; examples using the first data point –  $V = 11.89V$ ;  $d_s = 12.80cm$ ;  $d_1 = 0.2cm$ ;  $I_1 = 0.017A$ :

$$k_b = \frac{N_s \varphi_s}{n A d_s}$$

$$k_b = \frac{1.72mW}{(24)(\pi 0.0064m^2)(0.128m)}$$

$$k_b = 27.84693mWm^{-3}$$

$$\sigma_{d_s} = 0.00025m$$

$$\sigma_{k_b} = k_b \cdot \left( \frac{\sigma_{d_s}}{d_s} \right)$$

$$\sigma_{k_b} = 27.84693 \cdot \left( \frac{0.00025}{0.1280} \right)$$

$$\sigma_{k_b} = 0.05$$

$$k_b = 27.85 \pm 0.05mWm^{-3}$$

$$d_n = \sum_{m=1}^n d_m$$

$$\sigma_{d_n} = \sqrt{n(\sigma_{d_s})^2} = \sigma_{d_s} \sqrt{n}$$

$$\sigma_{d_1} = 0.00025m \cdot \sqrt{1}$$

$$\sigma_{d_1} = 0.00025m$$

$$B_n = k_b \cdot d_n$$

$$B_1 = 27.84693mWm^{-3} \times 0.002m$$

$$B_1 = 0.0557386mT$$

$$\sigma_{B_n} = B_n \sqrt{\left( \frac{\sigma_{k_b}}{k_b} \right)^2 + \left( \frac{\sigma_{d_n}}{d_n} \right)^2}$$

$$= k_b d_n \sqrt{\left( \frac{\sigma_{k_b}}{k_b} \right)^2 + \left( \frac{\sigma_{d_n}}{d_n} \right)^2}$$

$$\sigma_{B_1} = 27.84693mWm^{-3} \times 0.002m \sqrt{\left( \frac{0.05mWm^{-3}}{27.84693mWm^{-3}} \right)^2 + (1) \left( \frac{0.00025m}{0.1280m} \right)^2}$$

$$= 0.0001mT$$

$$B_1 = 0.0557 \pm 0.001mT$$



$$H = \frac{NI}{l}$$

$$H_1 = \frac{2385 \cdot 0.017A}{\pi 0.16m}$$

$$H_1 = 80.6617Am^{-1}$$

$$\sigma_I = 0.001A$$

$$\sigma_H = \frac{N}{l} \sigma_I$$

$$\sigma_H = \frac{2385}{\pi 0.16m} \cdot 0.001A$$

$$\sigma_H = 5Am^{-1}$$

$$H_1 = 81 \pm 5Am^{-1}$$

(H is in Ampere-turns per metre.)  
Computing the accumulated error on  $B_n$ :

$$\sigma_{B_n} = \sqrt{i\sigma_B^2}$$

$$\sigma_{B_1} = \sqrt{1 \cdot (0.00025m)^2}$$

$$= 0.00025m$$

The top and bottom curves of the hysteresis loop were fit to parametric Catmull-Rom splines. This is similar to connecting a line between each point, but instead a continuous cubic curve is smoothly interpolated between them. For some parameter  $s$  from zero to one used to interpolate  $p$  between two points,  $p_n$  and  $p_{n+1}$ , where  $p_n = \begin{pmatrix} H_n \\ B_n \end{pmatrix}$ .

Linear interpolation:

$$p = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} s \\ 1 \end{pmatrix} \cdot \begin{pmatrix} p_n \\ p_{n+1} \end{pmatrix}$$

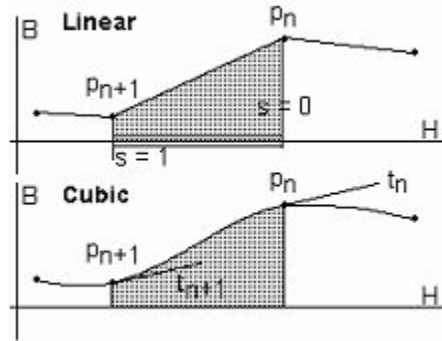
Hermite cubic interpolation that we used:

$$p = \begin{pmatrix} 2 & -3 & 0 & 1 \\ -2 & 3 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} s^3 \\ s^2 \\ s \\ 1 \end{pmatrix} \cdot \begin{pmatrix} p_n \\ p_{n+1} \\ t_n \\ t_{n+1} \end{pmatrix}$$

where  $t_1$  and  $t_2$  are the tangents at each point. These are calculated as:

$$t_n = \frac{1}{2}(p_{n+1} - p_{n-1})$$

except at the endpoints where  $p_n$  is replaced for the overflowing value and the result is doubled.  
For example, the curve between points 16 and 17 for trial one was parameterized as:



$$p_{16} = \begin{pmatrix} 2 & -3 & 0 & 1 \\ -2 & 3 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} s^3 \\ s^2 \\ s \\ 1 \end{pmatrix} \cdot \begin{pmatrix} \begin{pmatrix} 3795.845Am^{-1} \\ 2.589765mT \end{pmatrix} \\ \begin{pmatrix} 3103.104Am^{-1} \\ 2.53407mT \end{pmatrix} \\ \frac{1}{2} \begin{pmatrix} 2434.086Am^{-1} \\ 2.245053mT \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 4332.009Am^{-1} \\ 2.645459mT \end{pmatrix} = \begin{pmatrix} -614.452Am^{-1} \\ -0.05569mT \end{pmatrix} \\ \frac{1}{2} \begin{pmatrix} 3795.845Am^{-1} \\ 2.589765mT \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 2434.086Am^{-1} \\ 2.45053mT \end{pmatrix} = \begin{pmatrix} -680.88Am^{-1} \\ -0.06962mT \end{pmatrix} \end{pmatrix}$$

To begin calculating the area of between the curves, each segment of the spline is integrated:

$$\begin{aligned} A_n &= \int_0^1 H' B ds \\ &= \int_0^1 \left[ \frac{d}{ds} ((2s^3 - 3s^2 + 1)H_n + (-2s^3 + 3s^2)H_{n+1} + (s^3 - 2s^2 + s)tH_n + (s^3 - s^2)tH_{n+1}) \right] \\ &\quad ((2s^3 - 3s^2 + 1)B_n + (-2s^3 + 3s^2)B_{n+1} + (s^3 - 2s^2 + s)tB_n + (s^3 - s^2)tB_{n+1}) ds \\ &= \frac{1}{10} (-H_n t B_n + H_n t B_{n+1} + t H_n B_n - t H_n B_{n+1} + H_{n+1} t B_n - H_{n+1} t B_{n+1} - t H_{n+1} B_n + t H_{n+1} B_{n+1}) + \\ &\quad \frac{1}{2} (-H_n B_n - H_n B_{n+1} + H_{n+1} B_n + H_{n+1} B_{n+1}) + \\ &\quad \frac{1}{60} (t H_n t B_{n+1} - t H_{n+1} t B_n) \end{aligned}$$

For example, the segment 16 to 17 in trial one had an area of:

$$\begin{aligned} A_{16} &= \frac{1}{10} (-3595.845Am^{-1})(-0.05569mT) + (3595.845Am^{-1})(-0.06962mT) \\ &\quad + (-614.452Am^{-1})(2.589765mT) - (-614.452Am^{-1})(2.534071mT) \\ &\quad + (3103.104Am^{-1})(-0.05569mT) - (3103.104Am^{-1})(-0.06962mT) \\ &\quad - (-680.88Am^{-1})(2.589765mT) + (-680.88Am^{-1})(2.534071mT)) + \\ &\quad \frac{1}{2} (-3595.845Am^{-1})(2.589765mT) - (3595.845Am^{-1})(2.534071mT) \\ &\quad + (3103.104Am^{-1})(2.589765mT) + (3103.104Am^{-1})(2.534071mT)) + \\ &\quad \frac{1}{60} ((-614.452Am^{-1})(-0.06962mT) - (-680.88Am^{-1})(-0.05569mT)) \\ &= -1775.02mT Am^{-1} \end{aligned}$$

Because the endpoints of the curve don't line up, a line segment needs to be added to close the loop. This adds another integral to be computed. Note that the signs used in this calculation make this segment add with the same sign as the start of the top curve:

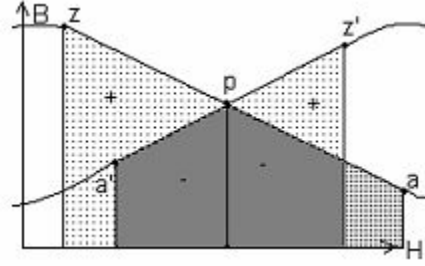


$$\begin{aligned}
A_t &= \int_{H_{bottom}}^{H_{top}} \left[ B_{bottom} + \frac{B_{top} - B_{bottom}}{H_{top} - H_{bottom}} (h - H_{bottom}) \right] dh \\
&= \frac{1}{2} (H_{top} - H_{bottom}) (B_{top} + B_{bottom})
\end{aligned}$$

For trial one:

$$\begin{aligned}
A_t &= \frac{1}{2} ((4332.009Am^{-1}) - (4469.608Am^{-1})) ((2.645459mT) + (2.81254mT)) \\
&= -375.5087mTAm^{-1}
\end{aligned}$$

This gives a closed curve, but some of our loops were not simple (it is they were self-intersecting.) Because we only consider positive area, the 'top' and 'bottom' of the curves must be swapped over intervals where they cross over. This leaves calculating the area for the interval on which the two curves intersect. Finding the intersection of two cubics then integrating between variable parameters turns out to be a very complicated calculation. Since part of the area on either side of the intersection will cancel out, the area of these intervals will be much smaller than that of other intervals. Doing linear interpolation for this one segment of the spline has no significant effect on the net result (doing linear interpolation on the entire curve has only a small effect.)



First, the indices of the points  $a$ ,  $a'$ ,  $z = a + 1$ , and  $z' = a' + 1$  on either side of the intersection are determined by inspection of the curve; next,  $H$  of the intersection point,  $p$ , is calculated:

$$H_p = \frac{B_{a'} - B_a + \left( \frac{B_z - B_a}{H_z - H_a} \right) H_a - \left( \frac{B_{z'} - B_{a'}}{H_{z'} - H_{a'}} \right) H_{a'}}{\left( \frac{B_z - B_a}{H_z - H_a} \right) - \left( \frac{B_{z'} - B_{a'}}{H_{z'} - H_{a'}} \right)}$$

For trial one,  $a$  was 18 and  $a'$  was 72:

$$\begin{aligned}
H_p &= \frac{B_{72} - B_{18} + \left( \frac{B_{19} - B_{18}}{H_{19} - H_{18}} \right) H_{18} - \left( \frac{B_{73} - B_{72}}{H_{73} - H_{72}} \right) H_{72}}{\left( \frac{B_{19} - B_{18}}{H_{19} - H_{18}} \right) - \left( \frac{B_{73} - B_{72}}{H_{73} - H_{72}} \right)} \\
&= \frac{(2.478377mT) - (2.45053mT) + \left( \frac{(2.394836mT) - (2.45053mT)}{(2040.267Am^{-1}) - (2434.086Am^{-1})} \right) (2434.086Am^{-1}) - \left( \frac{(2.617612mT) - (2.478377mT)}{(3112.593Am^{-1}) - (2448.32Am^{-1})} \right) (2448.32Am^{-1})}{\left( \frac{(2.394836mT) - (2.45053mT)}{(2040.267Am^{-1}) - (2434.086Am^{-1})} \right) - \left( \frac{(2.617612mT) - (2.478377mT)}{(3112.593Am^{-1}) - (2448.32Am^{-1})} \right)} \\
&= 2256.624Am^{-1}
\end{aligned}$$

Calculate the integral, swapping signs on either side of point  $p$ :

$$\begin{aligned}
A_I &= \int_{H_a}^{H_p} \left[ B_a + \frac{B_z - B_a}{H_z - H_a} (h - H_a) \right] dh - \int_{H_p}^{H_z} \left[ B_a + \frac{B_z - B_a}{H_z - H_a} (h - H_a) \right] dh - \int_{H_{a'}}^{H_p} \left[ B_{a'} + \frac{B_{z'} - B_{a'}}{H_{z'} - H_{a'}} (h - H_{a'}) \right] dh + \int_{H_p}^{H_{z'}} \left[ B_{a'} + \frac{B_{z'} - B_{a'}}{H_{z'} - H_{a'}} (h - H_{a'}) \right] dh \\
&= \frac{1}{2} (B_{a'} H_{z'} + B_{z'} H_{z'} + B_a H_z + B_z H_z - B_a H_a - B_z H_a - B_{a'} H_{a'} - B_{z'} H_{a'})
\end{aligned}$$

For trial one:

$$\begin{aligned}
A_I &= \frac{1}{2}((2.478377mT)(3112.593Am^{-1}) + (2.617612mT)(3112.593Am^{-1}) + (2.45053mT)(2040.267Am^{-1}) + (2.394836mT)(2040.267Am^{-1})) \\
&= 22.99033mTAm^{-1}
\end{aligned}$$

Note that integrals along the top curve give negative results because we are integrating from a higher  $H$  to a lower  $H$ . With this in mind, the final area function is given by summing the integrals under each segment and the closing segment. If the curve intersects, the two integrals  $A_a$  and  $A'_a$  are replaced by the area for the intersecting segment,  $A_I$ :

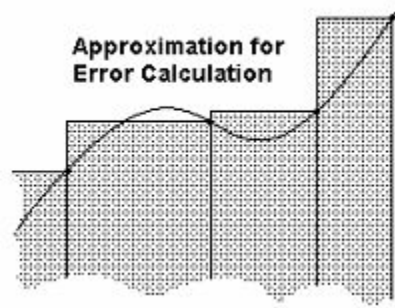
$$\oint_{loop} DH \cdot DB = \begin{cases} -\sum_{i=1}^{\text{end}} A_i - A_t, & \text{simple} \\ \sum_{i=1}^{a-1} A_i \sum_{i=a+1}^{a'-1} A_i \sum_{i=d+1}^{\text{end}} A_i + A_I + A_t, & \text{intersecting} \end{cases}$$

Since trial one was intersecting, its area was:

$$\begin{aligned}
\oint_{loop} DH \cdot DB &= (-1403.27mTAm^{-1} - 1775.02mTAm^{-1} - 1667.97mTAm^{-1}) - (-864.89mTAm^{-1} + \dots + 839.90mTAm^{-1}) \\
&= 4257.42mTAm^{-1}
\end{aligned}$$

The error on the area is calculated by first finding the errors on each term of the area sum. Using the method of partial derivatives on these formulae directly yields error equations several scores of printed pages in length for each term. Because this is an error calculation, approximations are definitely appropriate. For finding the error on the integral between each point, we use the following approximation:

$$A_n \int_{H_n}^{H_{n+1}} hB_n dh = B_n(H_{n+1} - H_n)$$



This gives the following error calculation (the error on  $H$  is constant for each trial):

$$\begin{aligned}
\sigma_{A_n} &= \sqrt{\left(\sigma_{B_n} \frac{A}{B_n}\right)^2 + \left(\sigma_H \frac{A}{H_n}\right)^2 + \left(\sigma_H \frac{A}{H_{n+1}}\right)^2} \\
&= \sqrt{2\sigma_H^2 \sigma_{B_n}^2 + \sigma_{B_n}^2 (H_{n+1} - H_n)^2}
\end{aligned}$$

In trial one, for example, one such error was:

$$\begin{aligned}
\sigma_{A_{16}} &= \sqrt{2(4.744807Am^{-1})^2(0.028303mT)^2 + (0.028303mT)^2((3103.104Am^{-1}) - (3795.845Am^{-1}))^2} \\
&= 26.19918mTAm^{-1}
\end{aligned}$$

The total error on the area is computed by taking the error on the sum of the integrals of each segment. Error on the closing and intersecting segments is ignored because they make up a very small portion of the total area (which is compensated for by including error on the segments they would have replaced.) The final approximation is well within 5% of the error that would have been obtained without taking any approximations:

$$\sigma_A = \sqrt{\sum \sigma_{An}^2}$$

For trial one this was:

$$\sigma_A = \sqrt{(23.06029Am^{-1})^2 + \dots + (40.34052Am^{-1})^2}$$

$$165.2311mTAm^{-1}$$

Work Calculations - volume:

$$V = 2\pi^2 R^2 r \quad = (\text{cross-sectional area} \cdot \text{circumference})$$

$$= (1 \times 10^{-4}m^2) \cdot (0.16m\pi)$$

$$= 5.02655 \times 10^{-5}m^3$$

Work (where  $V$  is volume):

$$W = V \cdot \text{area of hysteresis loop}$$

$$= (5.02655 \times 10^{-5}m^3)(4.25742AWbm^{-1})$$

$$= 2.14 \times 10^{-4}J$$

Error on work:

$$\sigma_W = W \cdot \sqrt{\left(\frac{\sigma_{\text{area}}}{\text{area}}\right)^2 + \left(\frac{\sigma_V}{V}\right)^2}$$

$$= 2.14 \times 10^{-4}J \cdot \sqrt{\left(\frac{0.165231AWbm^{-1}}{4.25742AWbm^{-1}}\right)^2 + \left(\frac{0m^3}{5.02655 \times 10^{-5}m^3}\right)^2}$$

$$= 8 \times 10^{-6}J$$

Steinmetz constant for each trial:

$$\eta_n = \frac{\text{area}}{B_{\text{max}}^{1.6}}$$

$$\eta_1 = \frac{4.25742AWbm^{-1}}{(0.002645T)^{1.6}}$$

$$= 56642.824$$

Individual error on Steinmetz constant:

$$\sigma_{\eta_n} = \eta_n \cdot \sqrt{\left(\frac{\sigma_{\text{area}}}{\text{area}}\right)^2 + \left(\frac{\sigma_{B_{\text{max}}}}{B_{\text{max}}}\right)^2}$$

$$\sigma_{\eta_1} = 56642.824 \cdot \sqrt{\left(\frac{0.165231AWbm^{-1}}{4.25742AWbm^{-1}}\right)^2 + \left(\frac{2.74533 \times 10^{-5}T}{0.0026455T}\right)^2}$$

$$= 2275.54783$$

Error of Steinmetz constant:

$$\begin{aligned}\sigma_{\eta} &= \sqrt{\sum \frac{1}{\sigma_{\eta n}^2}} \\ &= \sqrt{\frac{1}{(2275.547)^2} + \dots + \frac{1}{(1835.748)^2}} \\ &= 535.665\end{aligned}$$

Weighted average of Steinmetz constant:

$$\begin{aligned}\eta &= \sigma_{\eta}^2 \sum \frac{\eta^n}{\sigma_{\eta n}^2} \\ &= (535.66)^2 \cdot \left( \frac{(56642.824)}{(2275.547)^2} + \dots + \frac{(58982.244)}{(1835.748)^2} \right) \\ &= 56783.740\end{aligned}$$